TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 347, Number 9, September 1995

A NOTE ON SINGULARITIES IN SEMILINEAR PROBLEMS

MOHAMMED GUEDDA AND MOKHTAR KIRANE

ABSTRACT. We consider the equation $\Delta u - \frac{1}{2}x.\Delta u - \frac{u}{q-1} + u^q = 0$, for $q > 1$. We study the isolated singularities and present a nonlinear technique, and give a complete classification.

1. INTRODUCTION

In this note we study the isolated singularities of the positive solutions of the following nonlinear equation:

$$\Delta u - \frac{1}{2}x.\nabla u - \frac{u}{q-1} + u^q = 0. \tag{1.1}$$

Let Ω be an open subset of $\mathbf{R}^N$, $N \geq 2$, containing 0, $\Omega' = \Omega\backslash\{0\}$ and $q > 1$. We are concerned with the following question:

If $u \in C^2(\Omega')$ is a positive solution of (1.1) in Ω', what can be said about $u(x)$ and about the equation as $|x| \to 0$?

It is well known [15] that (1.1) has no nontrivial globally bounded solution for $N = 1, 2$ or $N > 2$, $q \leq \frac{N+2}{N-2}$ except the constant solution $u = (\frac{1}{q-1})^{1/(q-1)}$.

If we look for a specific solutions of (1.1) under the form

$$u(r) = \alpha r^\beta,$$

then we get

$$\beta = -\frac{1}{q-1} \quad \text{and} \quad \alpha = \lambda_{N,q} = \left\{\frac{2}{q-1}\left(N - \frac{2q}{q-1}\right)\right\}^{1/(q-1)}$$

where it is clear that $\lambda_{N,q}$ only exists when $N > 2$ and $q > \frac{N}{N-2}$. It follows by substitution that $u(r) = \lambda_{N,q} r^\beta$ is also a solution of the Emden-Fowler equation

$$\Delta w + w^q = 0. \tag{1.2}$$

This equation has been intensively studied. When $N > 2$ two critical values $\frac{N}{N-2}$ and $\frac{N+2}{N-2}$ appear. The first studies in the radial case are due to Emden, then Fowler [3, 5, 6], Brezis and Lions [2] and Lions [22] (see also [18, 1]).

Let us briefly describe our results. We have two cases.

Received by the editors July 6, 1994 and, in revised form, October 21, 1994.
1991 *Mathematics Subject Classification*. Primary 35J65.
The first author was supported by LAMIFA F Amiens.

First case. $1 < q < \frac{N}{N-2}$. Then any positive solution of (1.1) in Ω' satisfies the equation $-\Delta u+\frac{1}{2}x.\nabla u+\frac{u}{q-1}-u^q = \alpha\delta_0$ in $D'(\Omega)$ with $\alpha \geq 0$. Furthermore if $\alpha = 0$, then the "singularity is removable".

Second case. $\frac{N}{N-2} < q < \frac{N+2}{N-2}$. Then for any positive radial solution of (1.1) in Ω' we have $\operatorname{Lim} r^{2/(q-1)}u(r) = l \in \{0, \lambda_{N,q}\}$ as $r \to 0$, where $r = |x|$. If $l = 0$ and $q > 2$, the "singularity is removable". Note that if $q \geq \frac{N}{N-2}$, any solution of (1.1) satisfies

$$-\Delta u + \frac{1}{2}x.\nabla u + \frac{u}{q-1} - u^q = 0, \quad \text{in } D'(\Omega).$$

2. Main results

Let $N > 2$ and set $\Omega = B_R(0) = \{x \in \mathbf{R}^N, |x| < R\}$, $R > 0$ and $\Omega' = \Omega\backslash\{0\}$. Our main result is the following:

Theorem 2.1. *Assume that* $u \in L^1_{\text{Loc}}(\Omega')$,

$$\Delta u - \frac{1}{2}x.\nabla u \in L^1_{\text{Loc}}(\Omega'), \quad \textit{in } D'(\Omega'), \tag{2.1}$$

$$u \geq 0, \qquad \Delta u - \frac{1}{2}x.\nabla u \leq au + f \textit{ a.e. in } \Omega, \tag{2.2}$$

where a *is a nonnegative constant and* $f \in L^1_{\text{Loc}}(\Omega)$. *Then*

$$u \in L^1_{\text{Loc}}(\Omega), \tag{2.3}$$

and there exists $h \in L^1_{\text{Loc}}(\Omega)$, *and* $\alpha \geq 0$ *such that*

$$-\Delta u + \frac{1}{2}x.\nabla u = h + \alpha\delta_0 \quad \textit{in } D'(\Omega). \tag{2.4}$$

For the proof, we use a nonlinear technique introduced by Serrin [23], [24] and a linear method by H. Brezis and P. L. Lions [2], [22].

The proof is divided into five steps.

Step 1. We claim that

$$x.\nabla u = r\frac{\partial u}{\partial r}$$

where $x = (r, \sigma)$, $r = |x|$.

Proof of the claim. Writing

$$x.\nabla u = \sum_{i=1}^{N} x_i \left(\frac{\partial u}{\partial r}\frac{\partial r}{\partial x_i} + \sum_{j=1}^{N-1}\frac{\partial u}{\partial \sigma_j}\frac{\partial \sigma_j}{\partial x_i}\right),$$

$$x.\nabla u = r\frac{\partial u(r, \sigma)}{\partial r} + \sum_{j=1}^{N-1}\frac{\partial u}{\partial \sigma_j}\sum_{i=1}^{N} x_i\frac{\partial \sigma_j}{\partial x_i},$$

as in [17], we have

$$\frac{\partial \sigma_j}{\partial x_i} = \begin{cases} \frac{x_i x_{j+1}}{r_j r_{j+1}^2} & \text{if } i \leq j, \\ -\frac{r_j}{r_{j+1}^2} & \text{if } i = j+1, \\ 0 & \text{if } i > j+1, \end{cases}$$

where $r_j^2 = x_1^2 + x_2^2 + \cdots + x_j^2$, $r_N^2 = r^2$. Thus

$$\sum_{i=1}^{N} x_i \frac{\partial \sigma_j}{\partial x_i} = 0.$$

And then

$$x.\nabla u = r\frac{\partial u}{\partial r}. \tag{2.5}$$

Step 2. $u \in L^1_{\text{loc}}(\Omega)$. We begin by considering the average

$$\overline{u}(r) = \frac{1}{\omega_{N-1}} \int_{S^{N-1}} u(r, \sigma)\, d\sigma, \qquad 0 < r < R,$$

where ω_{N-1} is the volume of the sphere S^{N-1}. It follows from (2.1), (2.2) and (2.5) that

$$\Delta\overline{u} - \frac{1}{2} r\overline{u}_r \in L^1_{\text{Loc}}(0, R), \qquad \overline{u} \geq 0, \tag{2.6}$$

$$\Delta\overline{u} - \frac{1}{2} r\overline{u}_r \leq a\overline{u} + \overline{f} \quad \text{on } (0, R), \tag{2.7}$$

and

$$\frac{1}{r^{N-1}}(r^{N-1}K(r)\overline{u}_r)_r \leq a\overline{u} + \overline{f}, \quad \text{for } r \in (0, R), \tag{2.8}$$

where $K(r) = e^{-r^2/4}$. In particular $\overline{u} \in C^1(0, R)$.

Let $R' < R$ be fixed. Integrating (2.8) over (r, R') we find as in [2] that

$$u \in L^1_{\text{loc}}(\Omega) \tag{2.9}$$

and

$$\overline{u}(r) \leq \frac{C}{r^{N-2}} + C. \tag{2.10}$$

Step 3. Let $g(x) = -\operatorname{div}(K(x)\nabla u)$ a.e. in Ω, where $K(x) = e^{-|x|^2/4}$. Then $g \in L^1_{\text{loc}}(\Omega)$ and for any $\varphi \in D(\Omega)$, $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ near $x = 0$,

$$\int_\Omega g\varphi^2\, dx \leq -\int_\Omega u.\operatorname{div}(K(x)\nabla\varphi^2)\, dx. \tag{2.11}$$

Proof of the step 3. From the definition of g we have

$$\int_\Omega g\psi\, dx = \int_\Omega K(x)\nabla u\nabla\psi\, dx, \tag{2.12}$$

for any $\psi \in W^{1,\infty}(\Omega')$ with compact support in Ω'. Set

$$P_k(t) = \begin{cases} 1 & \text{if } t > 1 + k, \\ t - k & \text{if } k \leq t \leq 1 + k, \\ 0 & \text{if } t < k \end{cases}$$

for any $t \geq 0$ and $k \geq 0$.

Let $0 < \rho < R$ and $\varepsilon < \frac{\rho}{2}$. Let $\varphi \in D(\Omega)$ be such that $0 \le \varphi \le 1$, $\varphi \equiv 1$ on $B_\rho = \{x \in \mathbf{R}^N, |x| < \rho\}$ and $\psi_\varepsilon = \eta_\varepsilon \varphi$ with $\eta_\varepsilon \in C^\infty(\Omega)$, $0 \le \eta_\varepsilon \le 1$, $\eta_\varepsilon \equiv 0$ if $|x| < \varepsilon$, $\eta_\varepsilon(x) \equiv 1$ if $|x| > 2\varepsilon$ and $|\nabla \eta_\varepsilon| \le \frac{c}{\varepsilon}$. We have, from (2.12),

$$\begin{aligned}\int_\Omega g(1 - P_k(u))\psi_\varepsilon^2\,dx + \int_{\{k<u<1+k\}} K(x)|\nabla u|^2\psi_\varepsilon^2\,dx \\ = \int_\Omega (1 - P_k(u))K\nabla u \nabla \psi_\varepsilon^2\,dx.\end{aligned} \tag{2.13}$$

Since

$$\sum_{k=0}^{n}(1 - P_k(t))^+ = (n+1-t)^+,$$

we have

$$\begin{aligned}\int_{\{u<n+1\}} g(n+1-u)^+\psi_\varepsilon^2\,dx + \int_{\{u<1+n\}} K(x)|\nabla u|^2\psi_\varepsilon^2\,dx \\ \le \int_{\{u<n+1\}} (n+1-u)K\nabla u\nabla\varphi^2\eta_\varepsilon^2\,dx \\ + 2\int_{\{u<1+n\}\cap B_{2\varepsilon}} (n+1-u)K(x)|\nabla u|\psi_\varepsilon|\nabla\eta_\varepsilon|\,dx.\end{aligned}$$

Now for any real $h > 0$, we have $n+1-u(x) > n+1\frac{h}{h+1}$ a.e. in $\{u < \frac{n+1}{h+1}\}$. Thus, dividing by $n+1$ and using the Hölder inequality, for any $\beta > 0$,

$$\begin{aligned}\frac{h}{h+1}\int_{\{u<\frac{n+1}{h+1}\}} g\psi_\varepsilon^2 + \frac{1}{n+1}\int_{\{u<1+n\}} K|\nabla u|^2\psi_\varepsilon^2 \\ \le \int_{\{u<1+n\}} \left(1 - \frac{u}{n+1}\right)K\nabla u\nabla(\varphi^2)\eta_\varepsilon^2 \\ + \beta^2\int_{\{u<1+n\}} K(x)|\nabla u|^2\psi_\varepsilon^2 + \beta^{-2}\int_{B_{2\varepsilon}} K(x)|\nabla\eta_\varepsilon|^2.\end{aligned}$$

Let $\beta^2 = \frac{1}{2(n+1)}$; then

$$\begin{aligned}\frac{h}{h+1}\int_{\{u<\frac{n+1}{h+1}\}} g\psi_\varepsilon^2 + \frac{1}{2(n+1)}\int_{\{u<1+n\}} K|\nabla u|^2\psi_\varepsilon^2 \\ \le \int_{\{u<1+n\}} \left(1 - \frac{u}{n+1}\right)K\nabla u\nabla(\varphi^2)\eta_\varepsilon^2 + 2(n+1)C\varepsilon^{N-2}.\end{aligned}$$

From Fatou's Lemma—which is valid since $g \ge -au - f \in L^2_{\mathrm{loc}}(\Omega)$—we deduce as $\varepsilon \to 0$, $n \to +\infty$, $h \to +\infty$ that $g\varphi^2 \in L^1(\Omega)$ and satisfies (2.11).

Step 4. Now since $u \in L^1(\Omega)$, we can define the distribution

$$T = -\operatorname{div}(K\nabla u) - g \quad \text{in } D'(\Omega).$$

Then as in [2], we have

$$T = \sum_{|p|\le m} c_p D^p \delta_0.$$

Let $\psi \in D(B_R)$ be any fixed function such that

$$(-1)^{|p|}D^p\psi(0) = c_p \quad \text{for every } |p| \le m$$

and

$$\psi_\varepsilon(x) = \psi\left(\frac{x}{\varepsilon}\right).$$

Then

$$\int_{B_R} u\,\mathrm{div}(K\nabla\psi_\varepsilon) = \int_{B_R} g\psi_\varepsilon + \sum_{|p|\le m} \frac{c_p^2}{\varepsilon^{|p|}}. \tag{2.14}$$

On the other hand we have

$$\left|\int_{B_R} u\,\mathrm{div}(K\nabla\psi_\varepsilon)\right| \le \frac{C}{\varepsilon^2}\int_{B_{R\varepsilon}} u\,dx + CR\int_{B_{R\varepsilon}} u\,dx$$

and therefore

$$\left|\int_{B_R} u\,\mathrm{div}(K\nabla\psi_\varepsilon)\right| \le \frac{C}{\varepsilon^2}\int_0^{R\varepsilon} \overline{u}r^{n-1}\,dr + CR\int_0^{R\varepsilon} \overline{u}r^{n-1}\,dr.$$

We deduce from (2.10) that $|\int_{B_R} u\,\mathrm{div}(K\nabla\psi_\varepsilon)| \le C$. Comparing this with (2.14) we conclude that $c_p = 0$ when $|p| \ge 1$.

Finally we choose $\eta \in D(\Omega)$, $0 \le \eta \le 1$, and $\eta \equiv 1$ near $x = 0$.

We have

$$\langle T, \eta^2\rangle = c_0 = \int_\Omega u\,\mathrm{div}(K\nabla\eta^2) - \int_\Omega g\eta^2,$$

and hence $c_0 \ge 0$ from (2.11).

Step 5. The end of the proof of Theorem 2.1.

Let $h(x) = g(x)e^{|x|/4}$ a.e. in Ω; then $h \in L^1_{\mathrm{loc}}(\Omega)$ and $-\Delta u + \frac{1}{2}x.\nabla u = h + c_0\delta_0$ in $D'(\Omega)$.

3. The subcritical case

Now we return to equation (1.1). Let μ be the fundamental harmonic function in $\mathbf{R}^N\backslash\{0\}$, $N > 2$, that is

$$\mu(x) = \frac{1}{N(N-2)\omega_N}|x|^{2-N}. \tag{3.1}$$

Theorem 3.1. *Let $N > 2$, $1 < q < \frac{N}{N-2}$. Let $u \in C^2(\Omega')$ be a nonnegative solution of equation*

$$\Delta u - \frac{1}{2}x.\nabla u - \frac{u}{q-1} + u^q = 0 \quad \text{in } \Omega' = \Omega\backslash\{0\}. \tag{3.2}$$

Then

(i) *either u can be extended to a smooth solution of* (3.2) *in* Ω, *or*
(ii) *there exists $\alpha > 0$ such that $\lim_{x\to 0}\frac{u(x)}{\mu(x)} = \alpha$ and u satisfies the equation*

$$-\Delta u + \frac{1}{2}x.\nabla u + \frac{u}{q-1} - u^q = \alpha\delta_0 \quad \text{in } D'(\Omega). \tag{3.3}$$

Proof. From (3.2) we have $\Delta u - \frac{1}{2}x.\nabla u \le \frac{u}{q-1}$; hence from Theorem 2.1 we have

$$-\Delta u + \frac{1}{2}x.\nabla u + \frac{u}{q-1} - u^q = \beta\delta_0, \qquad u \in L^1_{\mathrm{loc}}(\Omega),\ u^q \in L^1_{\mathrm{loc}}(\Omega).$$

Moreover equation (3.2) can be written in the form

$$\operatorname{div}(K\nabla u) + d(x)u = 0 \quad \text{in } \Omega' \tag{3.4}$$

where $d(x) = k(x)(u^{q-1} - \frac{1}{q-1})$. Since $q < \frac{N}{N-2}$ we can find an $\varepsilon > 0$ such that $d \in L^{N/(2-\varepsilon)}_{\text{loc}}(\Omega)$. We then deduce from [24] that if u is singular at 0 there exists a constant $C > 0$ such that

$$C\mu \le u \le \frac{1}{C}\mu \quad \text{near } 0. \tag{3.5}$$

As in Guedda-Veron [18], we prove by scaling that there is an $\alpha > 0$ such that

$$\lim_{x\to 0} \frac{u(x)}{\mu(x)} = \alpha;$$

we get that

$$\lim_{x\to 0} |x|^{N-1}\nabla u(x) = \frac{\alpha}{N\omega_N}\xi, \quad \text{where } \xi = \lim_{x\to 0}\frac{x}{|x|} \tag{3.6}$$

and then (3.3).

4. The super critical case

We still assume that $\Omega = B_R(0) = \{x \in \mathbf{R}^N,\, |x| < R\}$ and $\Omega' = \Omega\backslash\{0\}$. In this section we present some results concerning the isolated singularities of the positive solutions of (1.1), that is, of

$$-\Delta u + \frac{1}{2}x.\nabla u + \frac{u}{q-1} - u^q = 0 \quad \text{in } \Omega', \tag{4.1}$$

where

$$\frac{N}{N-2} < q < \frac{N+2}{N-2} \quad \text{and} \quad N > 2.$$

If we look for the solution of (3.2) under the form $u(x) = u(|x|) = \alpha|x|^\beta$, then we get

$$\beta = -\frac{1}{q-1} \quad \text{and} \quad \alpha = \lambda_{N,q} = \left(\frac{2}{q-1}\left(N - \frac{2q}{q-1}\right)\right)^{1/(q-1)}.$$

Theorem 4.1. *Let $N > 2$, $q \ge \frac{N}{N-2}$. Let $u \in C^2(\Omega')$ be a nonnegative solution of (4.1) in $D'(\Omega')$. Then we have $\alpha = 0$ in (3.3), i.e.*

$$-\Delta u + \frac{1}{2}x.\nabla u + \frac{u}{q-1} - u^q = 0 \quad \textit{in } D'(\Omega).$$

Theorem 4.2. *Under the assumptions of Theorem 4.1, if*

$$u^{(q-1)N/2} \in L^1_{\text{loc}}(\Omega), \tag{4.2}$$

then u can be extended to Ω as a solution of (4.1) in Ω.

The proofs of Theorems 4.1 and 4.2 are the same as in Guedda and Veron [18] and they are omitted.

The main result of this section is the following

Theorem 4.3. *Let $N > 2$ and $\frac{N}{N-2} < q < \frac{N+2}{N-2}$ and u be a positive radial solution of* (4.1).

Then we have

(i) *$r^{2/(q-1)}u(r)$ has a limit l as $r \to 0$ and $l \in \{0, \lambda_{N,q}\}$.*

(ii) *If $l = 0$ and $q > 2$, then u can be extended to Ω as a $C^2(\Omega)$ solution of* (4.1) *in Ω.*

We need the following

Lemma 4.1. *Let u be a positive radial solution of* (4.1) *in Ω'. Then $r^{2/(q-1)}u(r)$ is bounded as $r \to 0$.*

Proof. Let

$$t = \log(r^\gamma), \quad v(t) = r^{2/(q-1)}u(r), \quad \gamma = \frac{4}{q-1} + 2 - N > 0; \tag{4.5}$$

then (4.1) takes the equivalent form

$$\gamma^2(v_{tt} + v_t) + \frac{1}{2}\gamma e^{-2t/\gamma}v_t + \lambda v + v^q = 0 \tag{4.6}$$

where $\lambda = \frac{2}{q-1}(\frac{2}{q-1} + 2 - N) < 0$. Let

$$L(t) = \frac{1}{2}\gamma^2 v_t^2 + \lambda\frac{v^2}{2} + \frac{v^{q+1}}{q+1}, \qquad \text{for } t \in [0, +\infty[.$$

From (4.6) the function L is nonincreasing. Hence for any $t_0 \le t < +\infty$

$$\frac{1}{2}\gamma^2 v_t^2 + \lambda\frac{v^2}{2} + \frac{v^{q+1}}{q+1} \le L(t_0). \tag{4.7}$$

Hence v is bounded, as $q > 1$.

Lemma 4.2. *Suppose $\frac{N}{N-2} < q < \frac{N+2}{N-2}$. Let v be any solution of* (4.6). *Then $\lim_{t\to+\infty} v_t = 0$ and v has a limit l at $+\infty$ and*

$$l(\lambda + l^{q-1}) = 0. \tag{4.8}$$

Proof. Using (4.6), (4.7) and Lemma 4.1 we have

$$v,\ v_t,\ v_{tt} \text{ and } v_{ttt} \text{ are bounded}. \tag{4.9}$$

Hence $v_t \in L^2(t_0, +\infty)$. This implies $v_{tt} \in L^2(t_0, +\infty)$.

It follows from this and (4.9) that

$$\lim_{t\to+\infty} v_t = \lim_{t\to+\infty} v_{tt} = 0.$$

For any $t \ge \tau \ge t_0$, we have

$$|v(t) - v(\tau)| \le \|v_t\|_{L^2(t_0, +\infty)} \cdot \sqrt{(t-\tau)},$$

and hence v has a limit l at $+\infty$. From (4.6) we get (4.8).

Proof of Theorem 4.3. From Lemmas 4.1 and 4.2 $r^{2/(q-1)}u(r)$ has a limit l where $l = 0$ or $\lambda_{N,q}$.

Let v be as in (4.5). Assume that $l = 0$. The proof of (ii) is divided into two steps.

Step 1. Assume v decreases to 0 as t tends to $+\infty$; then

$$v(t) \leq Ce^{-t}, \tag{4.10}$$

for $t \geq 0$ and some $C > 0$. In fact, from (4.6) we have

$$\gamma^2(v_t + v) = \int_t^{+\infty} \frac{1}{2}\gamma e^{-2t/\gamma} v_s + \lambda v + v^q \, ds. \tag{4.11}$$

Since v and v_t are bounded, $\omega(t) = \frac{1}{2}\gamma e^{-2t/\gamma} v_s + \lambda v + v^q$ is integrable; then $\lambda v + v^q$ is integrable and then v is integrable. Since $v_t \leq 0$ and $\lambda < 0$, we have

$$\gamma^2(v_t + v) \leq \int_t^{+\infty} v^q \, ds \leq v^{q-1} \int_t^{+\infty} v \, ds;$$

then

$$v_t + v \leq Cv^{q-1}, \quad \text{for } t \text{ large}, \tag{4.12}$$

which implies, if $q > 2$, (4.10). Moreover, Theorem 4.2 implies that if v satisfies (4.10), then u can be extended to Ω as a regular solution of (1.1) in Ω.

Step 2. Assume that v is not asymptotically monotone. Then there exists a sequence $\{t_n\}$ such that $v_t(t_n) = 0$, $v(t_{2n})$ is local minimum and $v(t_{2n+1})$ is local maximum, and (from the equation)

$$0 < v(t_{2n}) < \lambda_{N,q} < v(t_{2n+1})$$

which is not possible since $\lim_{t\to+\infty} v(t) = 0$, which ends the proof.

REFERENCES

1. M. F. Bidaut-Veron, *Local and global behaviour of solutions of quasilinear equations of Emden-Fowler equations type*, Arch. Rational Mech. Anal. **107** (1989), 293–324.
2. H. Brezis and P. L. Lions, *A note on isolated singularities for linear elliptic equations*, J. Math. Anal. Appl. **7A** (1981), 263–266.
3. R. Emden, *Gaskugeln, Anwendungen der mechanischen Warmentheorie auf Kosmologie und meteorologische Probleme*, Chap. XII, Teubner, Leipzig, 1907.
4. M. Escobedo and O. Kavian, *Variational problems related to self-similar solutions of the heat equation*, Nonlinear Anal., Theory, Methods Appl. **11** (1987), 1103–1133.
5. R. H. Fowler, *The form near infinity of real continuous solutions of a certain differential equation of second order*, Quart. J. Math. **45** (1914), 289–350.
6. ——, *The solutions of Emden's and similar differential equations*, Monthly Notices Roy. Astronom. Soc. **91** (1920), 63–91.
7. ——, *Further studies on Emden's and similar differential equations*, Quart J. Math. **2** (1931), 259–288.
8. A. Friedman, *Remarks on nonlinear parabolic equations*, Applications of Nonlinear Partial Differential Equations in Mathematical Physics, Amer. Math. Soc., Providence, RI, 1965, pp. 3–23.
9. A. Friedman and J. B. McLeod, *Blow-up of positive solutions of semilinear heat equations*, Indiana Univ. Math. J. **34** (1985), 425–447.
10. A. Friedman, *Blow-up of solutions of non linear parabolic equations*, Nonlinear Diffusion Equations and their Equilibrium States, vol. 1 (W.-H. Ni et al., eds), Springer-Verlag, 1988, pp. 301–318.
11. H. Fujita, *On the blowing up of solutions to the Cauchy problem for* $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. IA **13** (1966), 109–124.

12. V. A. Galaktionov, S. P. Kurdymov and A. A. Samarskii, *Asymptotic stability of invariant solutions of nonlinear heat-condiction equations with sources*, translation of Differentsial'nye Uravneniya **20** (1981), 461–476.

13. V. A. Galaktionov, *Proof of the localization of unbounded solutions of the nonlinear parabolic equation*, Differentsial'nye Uravneniya **21** (1985), 15–23.

14. B. Gidas and J. Spruck, *Global and local behaviour of positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math. **34** (1981), 525–598.

15. Y. Giga and R. V. Kohn, *Asymptotically self-similar blow-up for semilinear heat equations*, Comm. Pure Appl. Math. **38** (1985), 297–319.

16. ———, *Characterizing blowup using similarity variables*, Indiana Univ. Math. J. **36** (1987), 1–40.

17. A. Gmira, *Comportements asymptotiques et singularités des solutions de problèmes quasi-linéaires*, Doctorat d'Etat, Université de Tours, 1989.

18. M. Guedda and L. Veron, *Local and global properties of solutions of quasilinear equations*, J. Differential Equations **76** (1988), 159–189.

19. M. Guedda, *Propriétés locales et globales des solutions d'équations quasilinéaires elliptiques*, Ph.D. thesis, Université de Tours, 1987.

20. R. Hadiji and M. Guedda, *Some remarks on Sobolev inequality* (to appear).

21. S. Kaplan, *On the growth of solutions of quasilinear parabolic equations*, Comm. Pure Appl. Math. **16** (1963), 305–330.

22. P. L. Lions, *Isolated singularities in semilinear problems*, J. Differential Equations **38** (1980), 441–450.

23. J. Serrin, *Local behaviour of solutions of quasilinear equations*, Acta Math. **111** (1964), 247–302.

24. ———, *Isolated singularities of solutions of quasilinear equations*, Acta Math. **113** (1965), 219–240.

25. L. Veron, *Singularities of some quasilinear equations*, Nonlinear Diffusion Equations and their Equilibrium States (W. M. Ni, L. A. Peletier, and J. Serrin, eds.), M.S.R.I. Publ., Springer-Verlag, New York and Berlin, 1988.

Université de Picardie Jules Verne, Faculté de Mathématiques et d'Informatique, 33, Rue Saint Leu, 80039 Amiens Cedex 01 France

E-mail address: `guedda@crihan.fr`

Université de Annaba, Institut de Mathématiques, B.P. 12 Annaba, 2300 Algérie